

# Parallelisms, hyperplanes, and related problems in the theory of Veronese Spaces

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## Abstract

We study parallelisms on Veronese spaces associated with affine spaces, determine hyperplanes in Veronese spaces associated with projective spaces, and analyse the geometry of parallelisms determined by these hyperplanes.

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## Introduction

The term ‘Veronese space’ refers, primarily (and historically), to the structure of prisms in a projective space with ‘double hyperplanes’ as the points after that it refers to an algebraic variety which represents this structure (cf. e.g. [1]) and as such it was generalized in recent decades, and its geometry was studied and developed (see e.g. [2], [5], [6], [7]).

A task to find a synthetic approach to (‘original’) Veronese spaces was undertaken by the group of geometers around G. Tallini in the 70-th’s and the results are presented in [3], [8], [9], [16]. The way in which Veronese spaces were presented in that papers, where the points are two-element sets with repetitions with the elements being projective points, turned out to be generalizable fruitfully to an abstract construction, which associates with an arbitrary partial linear space (or even more generally: with an arbitrary incidence structure) another partial linear space (incidence structure resp.); this construction and its basic properties were presented in [12]. The point universe of the constructed Veronese space  $\mathbf{V}_k(\mathfrak{M})$  consists of the  $k$ -element sets with repetitions with the elements in the universe of the underlying ‘starting’ structure  $\mathfrak{M}$  (cf. definitions (2), (1)).

The construction discussed belongs to the family of, informally speaking, ‘multiplying’ a given structure somehow in the spirit of ‘manifold theory’: the structure  $\mathbf{V}_k(\mathfrak{M})$  can be covered by a family of copies of  $\mathfrak{M}$  so as through each point of it there pass  $k$  copies of  $\mathfrak{M}$ . Any two distinct copies of  $\mathfrak{M}$  in this covering do not share more than a single point. The abstract schemas (in fact: partial linear spaces of particular type) of the coverings induced by the construction of a Veronese space were studied in more detail in [15].

In this paper we develop to some extent the theory of Veronese spaces associated with spaces with some additional structures (like a parallelism), we discuss possibilities to introduce parallelism in the Veronese spaces and related questions, all in a sense referring to a broad problem: “hyperplanes and parallelisms”. Two main approaches are discussed: defining the Veronese space associated with a structure with a parallelism (say: with an affine space), and determining a hyperplane in a Veronese space associated with a linear space that contains hyperplanes (say: with a projective space) and delete this hyperplane. In both cases we obtain a structure which can be covered by several copies of an affine space; in the first one there is no simple way to introduce a parallelism on the defined ‘Veronese product’ (2.1), in the second one the obtained reduct cannot be presented as a Veronese space (2.19). Nevertheless we could succeed in characterizing the hyperplanes in (‘classical’) Veronese spaces associated with projective spaces (2.7) and prove that (similarly to the case when a hyperplane is deleted from a projective space) from the obtained partial affine partial line space the underlying Veronese space can be recovered (2.17). This completes the set of our main results. To be more exact: we find also interesting examples of hyperplanes in Veronese spaces associated with polar spaces and we generalize our construction to this class also. Here and there, throughout the paper, we formulate several open problems. They are out of the scope of our main reasoning but they seem closely related to it.

## 1 Basics

### 1.1 Incidence structures, partial linear spaces

In the paper we consider incidence structures i.e. structures of the form  $\mathfrak{W} = \langle S, \mathcal{B}, \mathbf{I} \rangle$ , where  $\mathbf{I} \subset S \times \mathcal{B}$  and  $\mathcal{B} \neq \emptyset$ .

In the context of (general) incidence structures the binary joinability relation (adjacency relation)  $\sim = \sim_{\mathcal{B}}$  defined over an incidence structure  $\langle S, \mathcal{B}, \mathbf{I} \rangle$  plays a fundamental role:

$$a \sim b \iff (\exists B \in \mathcal{B})[a, b \mathbf{I} B].$$

An incidence structure is *connected* when the relation (non-oriented graph)  $\sim$  is connected.

An incidence structure as above is a *partial linear space* (a PLS) if any two distinct blocks, if  $\mathbf{I}$ -related to two points, coincide, and any block is on at least three points (in most parts, in the literature it is assumed that a block of a PLS has at least two points, but *in this paper* we need a stronger requirement). In that case the incidence structure  $\mathfrak{W} = \langle S, \mathcal{B}, \mathbf{I} \rangle$  is isomorphic to the structure  $\langle S, \mathcal{B}^{\mathbf{I}}, \in \rangle$ , where  $\mathcal{B}^{\mathbf{I}} = \{\{x \in S : x \mathbf{I} B\} : B \in \mathcal{B}\}$ ; this second approach to partial linear spaces will be preferred in the paper.

A *subspace* of a partial linear space is a set  $X$  of its points such that each line which meets  $X$  in at least two points is entirely contained in  $X$ . A subspace  $X$  is *strong* (or *singular*, or *linear*) if any two its points are adjacent.

A *linear space* is a partial linear space in which any two points are adjacent. Two classes of linear spaces are of primary interest in geometry: affine and projective spaces, and structures associated with them are of primary interest in this paper.

## 1.2 Veronese Spaces

Let  $X$  be a nonempty set, we write  $\mathfrak{h}_k(X)$  for the set of  $k$ -element sets with repetitions with elements in  $X$  i.e. the set of all the functions  $f: X \rightarrow \mathbb{N}$  such that  $|f| := \sum_{x \in X} f(x) = k$ . Then, directly by the definition,  $f \in \mathfrak{h}_{|f|}(X)$ . If  $f \in \mathfrak{h}_k(X)$  then the set  $\{x \in X: f(x) \neq 0\} =: \text{supp}(f)$  is finite and  $\sum_{x \in X} f(x) = \sum_{x \in \text{supp}(f)} f(x)$ ; in such a case we write  $f = \sum_{x \in \text{supp}(f)} f(x) \cdot x$ . Finally, set  $\mathfrak{w}_k(X) = \bigcup \{\mathfrak{h}_l(X): l < k\}$ .

Let  $\mathfrak{W} = \langle S, \mathcal{B}, \mathbf{l} \rangle$  be an incidence structure and  $k$  be an integer. We set

$$\mathcal{B}^{\oplus} = \{e + rx: x \mathbf{l} B\}: 0 < r \leq k, e \in \mathfrak{h}_{k-r}(S), B \in \mathcal{B}\}. \quad (1)$$

Then we define (cf. [12])

$$\mathbf{V}_k(\mathfrak{W}) := \langle \mathfrak{h}_k(S), \mathcal{B}^{\oplus}, \in \rangle. \quad (2)$$

The structure  $\mathbf{V}_k(\mathfrak{W})$  will be called the *( $k$ -th) Veronese space associated with  $\mathfrak{W}$*  (or *over  $\mathfrak{W}$* ). We also say that it is a Veronese space of level  $k$ .

Let us quote after [12], [4]... the following simple observations.

**FACT 1.1.** *Let  $\mathfrak{W} = \langle S, \mathcal{B}, \mathbf{l} \rangle$  be an incidence structure. Assume that  $\mathfrak{W}$  satisfies the extensionality principle:*

$$B_1, B_2 \in \mathcal{B} \wedge (\forall x)[x \mathbf{l} B_1 \iff x \mathbf{l} B_2] \implies B_1 = B_2. \quad (\text{Ext})$$

Let  $k', r$  be nonnegative integers,  $r \geq 1$ . Fix  $e \in \mathfrak{h}_{k'}(S)$  and define the map

$$\mu_r: \mathfrak{h}_k(S) \ni f \longmapsto rf \in \mathfrak{h}_{rk}(S), \quad (3)$$

$$\tau_e: \mathfrak{h}_k(S) \ni f \longmapsto e + f \in \mathfrak{h}_{k+k'}(S) \quad (4)$$

Then  $\mu_r$  is an embedding of  $\mathbf{V}_k(\mathfrak{W})$  into  $\mathbf{V}_{rk}(\mathfrak{W})$  and  $\tau_e$  is an embedding of  $\mathbf{V}_k(\mathfrak{W})$  into  $\mathbf{V}_{k'+k}(\mathfrak{W})$ .

Moreover, the identification

$$*: S \ni x \longmapsto 1 \cdot x (= x^*) \in \mathfrak{h}_1(S) \quad (5)$$

is an isomorphism of  $\mathfrak{W}$  and  $\mathbf{V}_1(\mathfrak{W})$

**FACT 1.2.** *The structure*

$$\mathbf{V}_k(S) := \mathbf{V}_k(\langle S, \{S\}, \in \rangle) \quad (6)$$

is a partial linear space.

The lines of  $\mathbf{V}_k(S)$  will be called *leaves of  $\mathbf{V}_k(\mathfrak{W})$* .

The following is a folklore (see [12], [4])

**PROPOSITION 1.3.** *A Veronese space associated with a partial linear space is a partial linear space;*

Recall after [12, Prop. 2.9] the following

**FACT 1.4.** *Let  $B$  be a block of the Veronese space  $\mathbf{V}_k(\mathfrak{W})$  associated with an incidence structure  $\mathfrak{W}$  and  $\mathbf{T}(B)$  be the leaf of  $\mathbf{V}_k(\mathfrak{W})$  which contains  $B$ . If a point  $e$  is collinear with at least three points on  $B$  then  $e \in \mathbf{T}(B)$ .*

Note that a leaf  $e + rS$  of a Veronese space can be identified with  $e \in \mathfrak{w}_k(S)$ . Indeed, a leaf of  $\mathbf{V}_k(\mathfrak{W})$  uniquely associated with  $e$  has the form  $e + (k - |e|)S$ .

Each leaf  $e + (k - |e|)S$  of  $\mathbf{V}_k(\mathfrak{W})$  is a subspace of  $\mathbf{V}_k(\mathfrak{W})$  isomorphic to  $\mathfrak{W}$  under the map  $x \longmapsto e + (k - |e|) \cdot x$ .

### 1.3 Veblen and Net Configurations in Veronese spaces

A *Veblen configuration* consists of four lines no three concurrent, and any two intersecting each the other. In a more explicit way we call a *Veblen Configuration* a 4-tuple of lines  $L_1, L_2, M_1, M_2$  and a point  $p$  such that  $p \mid L_1, L_2$ ,  $p \nmid M_1, M_2$ ,  $L_i \sim M_j$  for  $i, j = 1, 2$ , and  $M_1 \sim M_2$ . An *incomplete Veblen configuration* is the family as above, where the condition  $M_1 \sim M_2$  is not assumed.

A *Net configuration* consists of four lines  $L_1, L_2, L_3, L_4$  and four points  $p_1, p_2, p_3, p_4$  such that  $p_i \mid L_i, L_{i+1 \bmod 4}$ , and  $p_i \nmid p_{i+2 \bmod 4}$  for  $i = 1, 2, 3, 4$ . In plain words, it is a quadrangle in which diagonals do not exist. In the literature the condition  $L_i \nmid L_{i+2 \bmod 4}$  (= ‘opposite sides do not intersect’) is frequently added; in the configurations arising via afinizations such a requirement is too restrictive.

Together with these two configurations two configurational axiom are considered: the *Veblen condition* and the *Net axiom*. The Veblen condition states that every incomplete Veblen configuration closes i.e. if  $p, L_1, L_2, M_1, M_2$  is an incomplete Veblen configuration defined above then  $M_1 \sim M_2$ . The net axiom states that if  $L_1, L_2, L_3, L_4$  is a quadrangle defined above,  $M_1 \sim L_1, L_3$ ,  $M_2 \sim L_2, L_4$  then  $M_1 \sim M_2$ . A partial linear space which satisfies the Veblen condition is called *veblenian*. Explicit forms of the Veblen configurations contained in a Veronese space are shown in [12, Lem. 3.1]. Let us recall this characterization.

**FACT 1.5.** *Let  $\mathfrak{M}_0$  be a partial linear space and  $\mathfrak{M} = \mathbf{V}_k(\mathfrak{M}_0)$ . A Veblen configuration contained in  $\mathfrak{M}$  arises as a result of a natural embedding of one of the following figures*

- (i) *a Veblen configuration contained in  $\mathfrak{M}_0$*
- (ii) *the set  $\{a + m : a \in A\}$  for a 4-subset  $A$  of a line  $m$  of  $\mathfrak{M}_0$ ,*
- (iii) *the set  $\{a + m : a \in A\} \cup \{2m\}$  for a 3-subset  $A$  of a line  $m$  of  $\mathfrak{M}_0$ .*

From this one derives, in particular, that the Veblen axiom is not, generally, preserved under Veronese products. Let  $\kappa(\mathfrak{M}_0) = \kappa$  be the size of a line of  $\mathfrak{M}_0$ . Note that  $\mathbf{V}_k(\mathfrak{M}_0)$  contains Veblen subconfiguration of the form (ii) iff  $\kappa \geq 4$  and it contains Veblen subconfiguration of the form (iii) iff  $\kappa \geq 3$ . However, one particular case ‘behaves’ more regularly. With the classification of triangles in Veronese spaces given in [4, Fact 4.1] and standard computations we directly justify

**FACT 1.6.** *If a partial linear space  $\mathfrak{M}_0$  is veblenian then  $\mathbf{V}_2(\mathfrak{M}_0)$  is veblenian as well.*

Combining 1.5 and 1.4 we obtain

**LEMMA 1.7.** *Let  $p, L_1, L_2, M_1, M_2$  form an incomplete Veblen configuration in a Veronese space associated with a partial linear space  $\mathfrak{M}_0$ . Assume, moreover, that points in one of the following pairs  $(L_1 \cap M_1, L_2 \cap M_2)$   $(L_1 \cap M_2, L_2 \cap M_1)$  are collinear. Then the lines  $L_1, L_2, M_1, M_2$  are contained in a leaf. So, if  $\mathfrak{M}_0$  is Veblenian, this configuration closes.*

Analogously, explicit forms of a realization of the Net Configuration in a Veronese Space were also established in [12]. It follows that the Net Axiom is not, generally, preserved under Veronese ‘products’. In the sequel we shall concentrate on

a pretty special case of Veronese Spaces, namely on the structures  $\mathbf{V}_k(\mathfrak{M}_0)$  with  $k = 2$  (i.e. on those originally considered in the history). In this case more ‘regular’ figures appear. Clearly, if  $L_1, L_2, L_3, L_4$  form a quadrangle with  $T(L_1) = T(L_3)$  then the quadrangle in question is obtained by an embedding of a quadrangle in  $\mathfrak{M}_0$ . If  $\mathfrak{M}_0$  is a linear space (and this case is, primarily, studied in the paper) and  $T(L_1) = T(L_2)$  then such a quadrangle has a diagonal. Consequently, searching for a quadrangle without diagonals we can restrict ourselves to the case  $\neq (T(L_1), T(L_2), T(L_3), T(L_4))$ . Such a quadrangle without diagonals will be called *proper* (Note: this definition makes sense only in structures in which the notion of a ‘leaf’ was introduced). The following is just a matter of direct (though quite tidy) verification. Let  $\mathfrak{M}_0$  be a linear space and  $\mathfrak{M} = \mathbf{V}_2(\mathfrak{M}_0)$

**LEMMA 1.8.** *Let  $L_1, K_1, L_2, K_2$  be lines of  $\mathfrak{M}$  in pairwise distinct leaves. These lines yield a quadrangle without diagonals iff one of the following holds, up to permutations: of the pairs  $(L_1, L_2), (K_1, K_2)$ , and of lines in each of these pairs.*

- (i) *There are lines  $m, n$  of  $\mathfrak{M}_0$  and points  $a_1, b_1 \in n, a_2, b_2 \in m$  such that*

$$L_1 = a_1 + m, L_2 = b_1 + m, K_1 = a_2 + n, \text{ and } K_2 = b_2 + n.$$

- (ii) *There are three lines  $m, n, l$  of  $\mathfrak{M}_0$  and points  $a, b, c$  such that  $a, b \in n, a, c \in m, b, c \in l$  and*

$$K_1 = 2n, K_2 = c + n, L_1 = a + m, \text{ and } L_2 = b + l.$$

*The vertices of the respective quadrangles are  $a_1 + a_2, a_1 + b_2, a_2 + b_1, b_1 + b_2$  in (i), and  $2a, a + c, 2b, b + c$  in (ii).*

**LEMMA 1.9.** *Let  $L_1, L_2$  be opposite sides of a proper quadrangle in  $\mathfrak{M}$ . Let  $K$  be a line of  $\mathfrak{M}$  crossing  $L_1, L_2$  such that  $T(K) \neq T(L_1), T(L_2)$ . Then one of the following holds.*

- (i)  *$L_1 = a + m, L_2 = b + m$  for  $a \neq b$  and a line  $m$  of  $\mathfrak{M}_0$ ; and  $K = x + \overline{a, b}$  for some  $x \in m$  or  $K = 2m$ , when  $a, b \in m$ .*
- (ii)  *$L_1 = 2n, L_2 = c + n, K = x + \overline{x, c}$  for  $x \in n, x \neq c$ .*
- (iii)  *$L_1 = a + m_1, L_2 = b + m_2$  for  $a \neq b$  and distinct lines  $m_1, m_2$  of  $\mathfrak{M}_0$  that share a point  $c$ . Then  $K = 2\overline{a, b}$  or  $K = c + \overline{a, b}$ .*

Finally, gathering together the possibilities listed in 1.8 and 1.9 we conclude with

**PROPOSITION 1.10.** *If  $K_1, K_2$  are two lines of  $\mathfrak{M}$  which cross two other lines  $L_1, L_2$  so as  $L_1, K_1, L_2, K_2$  is a proper quadrangle,  $L_3$  crosses  $K_1, K_2$ , and  $K_3$  crosses  $L_1, L_2$  then  $K_3, L_3$  share a point.*

Loosely (and not really strictly) speaking:  $\mathbf{V}_2(\mathfrak{M}_0)$  satisfies the Net Axiom.

#### 1.4 ‘ver-gras’: Grassmannians within Veronesians

A construction of the *combinatorial Grassmann space*  $\mathbf{G}_k(S)$  and of the *dual combinatorial Grassmann Space*  $\mathbf{G}_k^*(S)$ , very similar to the construction of the structure of leaves (*combinatorial Veronesian*)  $\mathbf{V}_k(S)$ , was introduced in [14]. Let us write

$\wp(S)$  for the set of all subsets of a set  $S$  and  $\wp_k(S)$  for the set of  $k$ -subsets of  $S$  ( $k$ : a positive integer). The structure  $\mathbf{G}_k(S)$  is a partial linear space with the pointset  $\wp_k(S)$  and the lines of the form  $\wp_k(U)$ , where  $U \in \wp_{k+1}(S)$ . The pointset of  $\mathbf{G}_k^*(S)$  is also  $\wp_k(S)$ , and the line set consist of the sets  $\{U \in \wp_k(S) : U_0 \subset U\}$ , where  $U_0$  varies over  $\wp_{k-1}(S)$ . Clearly, a subset  $a \in \wp_k(S)$  can be identified with its characteristic function  $\chi_a: S \rightarrow \{0, 1\} \subset \mathbb{N}$  and then  $\wp_k(S)$  appears to be a subset of  $\eta_k(S)$ . Suppose that a family of blocks  $\mathcal{B}$  is distinguished over  $S$  so as an incidence structure  $\mathfrak{M} = \langle S, \mathcal{B} \rangle$  is given. We write  $\mathbf{G}_k(\mathfrak{M})$  for the restriction of  $\mathbf{V}_k(\mathfrak{M})$  to the point set  $\wp_k(S)$ .

**NOTE 1.** Comparing the lines of corresponding structures we see that  $\mathbf{G}_k(S)$  is not a restriction of  $\mathbf{V}_k(S)$  to  $\wp_k(S)$ . It is quite another problem (purely combinatorial) to either study restriction of  $\mathbf{V}_k(S)$  to  $\wp_k(S)$  or determine blocks on  $\eta_k(S)$  so as the restriction of the obtained structure will be  $\mathbf{G}_k(S)$ .

The structure defined above is highly inhomogeneous. Indeed, the following is immediate from the definition:

**FACT 1.11.** Let  $\mathfrak{M} = \langle S, \mathcal{B} \rangle$  be an incidence structure. Take  $B \in \mathcal{B}$  and  $a \in \wp(S)$ .

- (i) If a block  $a + rB$  of  $\mathbf{V}_k(\mathfrak{M})$  yields a block of  $\mathbf{G}_k(\mathfrak{M})$  then  $r = 1$ ; moreover, clearly  $|a| = k - 1$ .
- (ii) Let  $|a| = k - 1$ . Then

$$(a + 1B) \cap \wp_k(S) = \{a + x : x \in B, x \notin a\}. \quad (7)$$

Therefore  $|(a + B) \cap \wp_k(S)| = |B \setminus a| = |B| - |B \cap a|$  and, consequently, the size of blocks may vary from 1 to the size of blocks of  $\mathfrak{M}$ .

Nevertheless, there are close connections between the Grassmannians and Veroneseans.

**PROPOSITION 1.12.** Let  $\mathfrak{M} = \langle S, \mathcal{L} \rangle$  be a linear space and  $k$  be a positive integer. Assume that the size of every line of  $\mathfrak{M}$  is at least  $k + 2$ . Then  $\mathbf{G}_k^*(S)$  can be defined in terms of  $\mathbf{G}_k(\mathfrak{M})$

PROOF. Let  $L$  be a line of  $\mathfrak{M}$  and  $q \in \wp_{k-1}(S)$ , assume that  $q + L$  is a block of  $\mathbf{G}_k(\mathfrak{M})$ . Consider the set

$$S(q + L) := \left\{ a \in \wp_k(S) : |\{p : (q + L) \ni p \sim_{\mathbf{G}_k(\mathfrak{M})} a\}| \geq 3 \right\}. \quad (8)$$

Then we prove that

$$S(q + L) = \{a \in \wp_k(S) : q \subset a\}. \quad (9)$$

Indeed. From 1.4 we get that  $S(q + L) \subset T(q + L) = q + S$ . Let  $a \in q + S \cap \wp_k(S)$ ; this means  $a = q \cup \{x\}$  with some  $x \notin a$ . At most  $k - 1$  lines of  $\mathfrak{M}$  of the form  $\overline{xy}$  with  $y \in L$  pass through a point in  $q$ , so there are at least 3 of them which miss  $q$ . Consequently,  $a$  is collinear with at least 3 points on  $q + L$ ; in accordance with definition,  $a \in S(q + L)$ .

As sets at the right-hand side of (9) are exactly the lines of  $\mathbf{G}_k^*(S)$ , we are through.  $\square$

**PROBLEM 1.13.** Is it possible to define  $\mathbf{V}_k(\mathfrak{M})$  in terms of  $\mathbf{G}_k(\mathfrak{M})$ . When the Grassmannian in question is further restricted to a complement of a hyperplane, the answer seems positive, as the missing points are exactly the directions, but without a parallelism, it may be hard!  $\circ$

## 2 Problems, their solutions, and conjectures

### 2.1 Multiplying a parallelism

Let  $\langle S_0, \mathcal{L}_0 \rangle$  be a partial linear space. A relation  $\parallel \subset \mathcal{L}_0 \times \mathcal{L}_0$  is called a *partial parallelism* (a *preparallelism*) if it is an equivalence relation such that distinct parallel lines are disjoint. If  $\parallel$  is a preparallelism as above then the structure  $\langle S_0, \mathcal{L}_0, \parallel \rangle$  is called a *partial affine partial line space*; the relation  $\parallel$  is called a *parallelism* and the respective structure is called an *affine partial line space* when the following form of the (affine) Euclid axiom holds: *the equivalence class  $[L]_\parallel$  (the direction) of each line  $L$  covers  $S_0$* . The most celebrated class of affine partial line spaces constitute affine spaces. Recall that the natural parallelism of an affine space coincides with the relation (so-called *Veblen parallelism*)  $\parallel^\circ$  defined by the formula

$$\begin{aligned} L_1 \parallel^\circ L_2 \iff & L_1 = L_2 \vee (L_1 \not\sim L_2 \\ & \& \text{ there are two lines } L', L'' \text{ through a point } p \\ & \text{ such that } p \nmid L_1, L_2 \wedge L_1, L_2 \sim L', L' \\ & \& \text{ there are collinear points } a_1 \in L_1 \cap L', a_2 \in L_2 \cap L''). \end{aligned} \quad (10)$$

Intuitively speaking:  $L_1 \parallel^\circ L_2$  when  $L_1, L_2$  are on a plane and either coincide or have no common point.

In most parts (e.g. in the case of affine spaces and their Segre products, cf. [13]) (10) is equivalent to a simpler formula with the condition *there are collinear points*  $a_1 \in L_1 \cap L', a_2 \in L_2 \cap L''$  on its right-hand side omitted. Here, we must handle with this more complex formula since in a Veronese space not every triangle determines a plane.

The first group of problems can be summarized in the following question: *how to extend a parallelism from a given structure to a Veronese space associated with it*. Let us start with the following approach, which may seem, at the first look, natural (especially taking into account similarities between Veronese and Segre products).

Let  $\mathfrak{A}_0 = \langle S_0, \mathcal{L}_0, \parallel_0 \rangle$  be a partial affine partial line space. We define on the lines of  $\mathbf{V}_k(\langle S_0, \mathcal{L}_0 \rangle)$  the relation  $\parallel$  by the formula

$$\begin{aligned} B_1 \parallel B_2 \iff & \text{ there are } e_1, r_1, r_2, L_1, L_2 \in \mathcal{L}_0 \text{ s.t.} \\ & B_1 = e_1 + r_1 L_1 \wedge B_2 = e_2 + r_2 L_2 \wedge L_1 \parallel L_2. \end{aligned} \quad (11)$$

Then we set

$$\mathbf{V}_k(\mathfrak{A}_0) = \langle \mathfrak{h}_k(S), \mathcal{L}^{\mathbb{B}}, \parallel \rangle.$$

**Remark 1.** Let  $\mathfrak{A}_0$  be an affine partial line space,  $k > 1$ , and  $\mathfrak{A} = \mathbf{V}_k(\mathfrak{A}_0)$ . Then  $\parallel$  defined by (11) is an equivalence relation and each of its equivalence classes covers

the point set of  $\mathfrak{A}$ . If a line  $L$  of  $\mathfrak{A}$  is given then in each of the leaves through a point  $f$  there passes exactly one line parallel to  $L$ .

Since there are  $k$  leaves through  $f$ , the relation  $\parallel$  is *not* a parallelism in  $\langle \mathfrak{h}_k(S), \mathcal{L}^{\mathfrak{B}} \rangle$  and therefore  $\mathfrak{A} = \mathbf{V}_k(\mathfrak{A}_0)$  is *not* a partial affine partial line space.

PROOF. It is evident that  $\parallel$  defined by (11) is an equivalence relation in  $\mathcal{L}^{\mathfrak{B}}$ . To justify the ‘negative’ part note, firstly, that an equivalence class  $[e + rL]_{\parallel}$  is determined uniquely by the line  $L$  of  $\mathfrak{A}_0$ , and each leaf is isomorphic to  $\mathfrak{A}_0$ .  $\square$

In essence, the situation is a bit more complex. Let us begin with a few words on the general theory of “imposing a parallelism onto a (finite) partial linear space”. So, assume that  $\mathfrak{A} = \langle X, \mathcal{L}, \parallel \rangle$  is a finite affine partial line space with fixed number  $r = r_{\mathfrak{A}}$  of lines through a point and fixed size  $\kappa = \kappa_{\mathfrak{A}}$  of each of its lines. Write  $v = v_{\mathfrak{A}} = |X|$  and  $b = b_{\mathfrak{A}} = |\mathcal{L}|$ . Moreover, assume that each direction  $[L_0]_{\parallel} = \{L \in \mathcal{L} : L \parallel L_0\}$  of a line  $L_0 \in \mathcal{L}$  has the size  $\delta = \delta_{\mathfrak{A}}$ . Then the following equations must be satisfied

$$v \cdot r = b \cdot \kappa, \quad (12)$$

$$\delta \cdot \kappa = v, \quad (13)$$

(12) is the so called fundamental equation of the theory of partial linear spaces, (13) is obtained when we observe that a direction must cover the point set of  $\mathfrak{A}$ .

If  $\mathfrak{M}_0$  is a configuration on  $v_0$  points and  $b_0$  lines with the corresponding point- and line-ranks  $r_0$  and  $\kappa_0$  then the parameters of  $\mathfrak{M} = \mathbf{V}_k(\mathfrak{M}_0)$  are as follows:  $v_{\mathfrak{M}} = \binom{v_0+k-1}{k}$ ,  $r_{\mathfrak{M}} = k \cdot r_0$ ,  $\kappa_{\mathfrak{M}} = \kappa_0$ , and  $b_{\mathfrak{M}} = \binom{v_0+k-1}{k-1} \cdot b_0$ .

**THEOREM 2.1.** *Let  $k > 1$ . There is no finite affine partial line space  $\mathfrak{A}_0$  with the fixed size of directions of its lines such that  $\mathfrak{A} = \mathbf{V}_k(\mathfrak{A}_0)$  admits a parallelism, the directions of lines of  $\mathfrak{A}$  have a constant size, and each leaf of  $\mathfrak{A}$  is an affine subspace of  $\mathfrak{A}$  (i.e. it each leaf is closed under this parallelism).*

PROOF. Suppose, to the contrary, that there are such  $\mathfrak{A}$  and  $\mathfrak{A}_0$ ; let  $n$  be the number of points of  $\mathfrak{A}_0$ . Denote by  $\delta_0 = \delta_{\mathfrak{A}_0}$  and  $\delta = \delta_{\mathfrak{A}}$  the corresponding sizes of directions. Let  $\kappa$  be the line size of  $\mathfrak{A}_0$ , and thus of  $\mathfrak{A}$  as well. From (13) we get

$$\delta_0 \cdot \kappa = n \text{ and } \delta \cdot \kappa = \binom{n+k-1}{k}.$$

Since no two leaves of  $\mathfrak{A}$  have a line in common, a direction of  $\mathfrak{A}$  is a union of directions considered in each of the leaves, and each of these leaves is isomorphic to  $\mathfrak{A}_0$ . This gives us the relation  $\delta = \text{the number of leaves} \cdot \delta_0$ , which yields  $\binom{n+k-1}{k} = n \binom{n+k-1}{k-1}$ , so  $k = 1$ .  $\square$

In particular, 2.1 yields a rather strange result: the structure  $\mathbf{V}_k(AG(m, q))$  can be covered by a family of subspaces each one isomorphic to the affine space  $AG(m, q)$  and therefore a natural parallelism intrinsically definable exists in each of the covering subspaces:

**Remark 2.** Let  $\mathfrak{A}_0$  be an affine space and let  $L_1, L_2$  be lines of  $\mathbf{V}_k(\mathfrak{A}_0)$ . Then  $L_1 \parallel^\circ L_2$  iff there are lines  $l_1, l_2$  parallel in  $\mathfrak{A}_0$  such that  $L_i = e + (k - |e|)l_i$  for some  $e$  and  $i = 1, 2$ .



(the proof is immediate after (10) and 1.4). But the union of the parallelisms in leaves is a partial parallelism and there is no ‘global’ parallelism definable on the whole line set of  $\mathbf{V}_k(AG(m, q))$ .

However, theorem 2.1 does not mean that  $\mathbf{V}_k(\mathfrak{A}_0)$  does not admit *some* parallelism: but this parallelism cannot agree in a natural way with the parallelisms on leaves. A search for such parallelisms is not the subject of this paper. Note, however, some limitations. Namely, with the techniques of the proof of 2.1 one can compute that if parallelisms required exist with constant sizes of the directions then (notation as above)

$$\delta/\delta_0 = \frac{1}{n+k} \binom{n+k}{k}, \text{ in case } k = 2: \delta/\delta_0 = \frac{n+1}{2}.$$

When  $\mathfrak{A}_0$  is a resolvable Steiner triple system from the Kirkmann Criterion, and directly when  $\mathfrak{A}_0$  is a finite affine space  $AG(m, q)$ ,  $q$  odd, we deduce that  $n$  is even and then  $\delta_{\mathbf{V}_2(\mathfrak{A}_0)}/\delta_{\mathfrak{A}_0}$  is an integer.

## 2.2 Hyperplanes in Veronese spaces

A second group of problems concentrates around ‘affinizations’ of Veronese structures: these problems concern the question *what are hyperplanes in Veronese structures, if they exist*. A set  $X$  of points of a partial linear space  $\mathfrak{M} = \langle S, \mathcal{L} \rangle$  is *l-transversal* if it meets every line of  $\mathfrak{M}$ . Clearly,  $S$  is l-transversal; a proper l-transversal subspace is called a *hyperplane*.

A subspace  $X$  of a partial linear space is called *spiky* when through each point on  $X$  there goes a line that is not contained in  $X$  (so, it meets  $X$  in a given point only), and  $X$  is *flappy* when through each line contained in  $X$  there passes a plane not contained in  $X$ .

A hyperplane  $\mathcal{H}$  of a partial linear space  $\mathfrak{M}$  determines a parallelism  $\parallel_{\mathcal{H}}$  on the lines not contained in  $\mathcal{H}$  defined by the formula

$$L_1 \parallel_{\mathcal{H}} L_2 \iff L_1 \cap \mathcal{H} = L_2 \cap \mathcal{H}.$$

Set  $\mathcal{L}^\times = \{L \setminus \mathcal{H} : L \in \mathcal{L}, L \not\subset \mathcal{H}\}$ . Recall that we have assumed  $|L| \geq 3$  for every  $L \in \mathcal{L}$ . Then each  $l \in \mathcal{L}^\times$  uniquely determines  $\bar{l} \in \mathcal{L}$  such that  $l \subset \bar{l}$  and it makes sense to define the parallelism  $\parallel_{\mathcal{H}}$  on  $\mathcal{L}^\times$  by the condition  $l_1 \parallel_{\mathcal{H}} l_2 \iff \bar{l}_1 \parallel_{\mathcal{H}} \bar{l}_2$ . Let us write

$$\mathfrak{M} \setminus \mathcal{H} = \langle S \setminus \mathcal{H}, \mathcal{L}^\times, \parallel_{\mathcal{H}} \rangle.$$

Then  $\mathfrak{M} \setminus \mathcal{H}$  is a partial affine partial line space. Note that it is not necessarily an affine partial line space. If  $\mathcal{H}$  is spiky then the points of  $\mathcal{H}$  can be interpreted in terms of  $\mathfrak{M} \setminus \mathcal{H}$  as the equivalence classes of the parallelism  $\parallel_{\mathcal{H}}$ . (comp. definitions in [13]).

Let  $\mathfrak{M}_0 = \langle S, \mathcal{L}_0 \rangle$  be a partial linear space; let  $\mathfrak{M} = \mathbf{V}_k(\mathfrak{M}_0)$  and  $\mathfrak{M}^* = \mathbf{V}_k(S)$  be the structure of the leaves of  $\mathfrak{M}$ . Let us pass to our main goal: determine the hyperplanes in  $\mathfrak{M}$ .

The first solution that comes to mind, though natural, is defective:

**Remark 3.** Let  $\mathcal{H}_0$  be a hyperplane of  $\mathfrak{M}_0$ . Then, clearly, the set  $\mathfrak{H}_k(\mathcal{H}_0)$  is a subspace of  $\mathfrak{M}$  but it is not l-transversal for  $k > 1$ .

PROOF. Let  $L \in \mathcal{L}_0$  be not contained in  $\mathcal{H}_0$ , and  $b \in L$  such that  $b \notin \mathcal{H}_0$ . Then  $(b + L) \cap \mathfrak{H}_k(\mathcal{H}_0) = \emptyset$ .  $\square$

Let  $\mathcal{H}$  be a hyperplane of  $\mathfrak{M}$ . Then, for each leaf  $S' = e + (k - |e|)S$  the intersection  $S' \cap \mathcal{H}$  is an  $\mathfrak{l}$ -transversal subspace of  $S'$ , so it is determined by an  $\mathfrak{l}$ -transversal subspace  $\mathcal{H}_e$  of  $\mathfrak{M}_0$ . Write  $\mathcal{H}$  for the set of all the hyperplanes of  $\mathfrak{M}_0$ . So,  $\mathcal{H}$  determines via the formula  $\mathfrak{h}(e) = \{x \in S : e + (k - |e|)x \in \mathcal{H}\}$  a function

$$\mathfrak{h} : \mathfrak{w}_k(S) \longrightarrow \mathcal{H} \cup \{S\} \quad (14)$$

such that

$$\mathcal{H} = \bigcup \{e + (k - |e|)\mathfrak{h}_e : e \in \mathfrak{w}_k(S)\} \quad (\text{we write } \mathfrak{h}_e = \mathfrak{h}(e)). \quad (15)$$

Moreover,  $\mathcal{H}$  is an  $\mathfrak{l}$ -transversal set in  $\mathfrak{M}^*$ . The following is a standard exercise.

**LEMMA 2.2.** *For every function as in (14) the set  $\mathcal{H}$  defined by (15) is  $\mathfrak{l}$ -transversal in  $\mathfrak{M}$ .*

**PROBLEM 2.3.** *What properties should, generally, a function  $\mathfrak{h}$  meet so as  $\mathcal{H}$  defined by (15) is a subspace and, consequently, it is a hyperplane? In a less general settings, it suffices to find what general properties of a function  $\mathfrak{h}$  assure that with the above definitions adopted we get*

$$\left( \bigcup \{e + (k - |e|)\mathfrak{h}_e : e \in \mathfrak{w}_k(S)\} \right) \cap (e' + (k - |e'|)S) = e' + (k - |e'|)\mathfrak{h}_{e'}$$

for each  $e' \in \mathfrak{w}_k(S)$ . Clearly, the latter condition yields that  $\mathcal{H}$  is a subspace.  $\circ$

One can give some “combinatorial” answers to the above problems, but it is not the subject of the paper to analyze the obtained solutions in their full generality. Instead, in what follows we shall give a series of interesting (we believe) examples of hyperplanes in Veronese spaces associated with ‘classical’ geometries.

Let us recall at the end of this part properties characteristic for projective and affine spaces. The projective spaces are the veblenian linear spaces. Affine spaces satisfy the *Tamaschke Bedingung* (if a line parallel to one side of a triangle crosses a second side then it crosses the third side as well) and *Parallelogram Completion Condition* (if of two pairs of parallel lines three intersections of lines in pairs of non parallel lines exist, then the fourth intersection point exists as well).

### 2.2.1 First example: hyperplanes in Veronese spaces associated with projective spaces

Let  $\varkappa$  be a (possibly degenerate) correlation in a projective space  $\mathfrak{P} = \langle S, \mathcal{L} \rangle$  over a field with odd characteristic. That means, there is a non zero reflexive form  $\xi$  on the vector space  $\mathbb{V}$  coordinatizing  $\mathfrak{P}$  such that  $\langle u \rangle \in \varkappa(\langle v \rangle)$  is equivalent to  $\xi(u, v) = 0$  for any non zero vectors  $u, v$  of  $\mathbb{V}$ . Consider  $\mathfrak{M} = \mathbf{V}_2(\mathfrak{P})$  (the primary example of a Veronese space: the Veronese variety, cf. [9]) and define a function  $\mathfrak{h}$  on  $\mathfrak{w}_2(S)$  as required in (14). Firstly, we set

$$\mathfrak{h}(x) = \varkappa(x) \quad \text{for } x \in S \quad (16)$$

In accordance with (14), either  $\mathfrak{h}(0) = S$  or  $\mathfrak{h}(0) = h_0$  for some hyperplane  $h_0$  of  $\mathfrak{P}$ .

**LEMMA 2.4.** *Suppose that  $\mathfrak{h}(0) = h_0$  and assume that  $h_0 \not\subset \{a : a \in \varkappa(a)\}$ . Let  $\mathcal{H}$  be defined by (15). Then  $\mathcal{H}$  is not a subspace of  $\mathfrak{M}$ .*

PROOF. Let  $a \in h_0$ ,  $a \notin \varkappa(a)$ , and let  $q \in \varkappa(a) \setminus h_0$ . Then  $2a, a + q \in a + \overline{a, q} \cap \mathcal{H}$ . Suppose  $a + \overline{a, q} \subset \mathcal{H}$ , then there exists a point  $q' \in \overline{a, q}$ ,  $q \neq q'$  with  $q' \in \varkappa(a)$ , so (contradictory)  $a \in \varkappa(a)$ .  $\square$

Accordingly, to ‘extend’  $\varkappa$  to a function  $\mathfrak{h}$  that determines a hyperplane in  $\mathfrak{M}$  we must put

$$\mathfrak{h}(0) = S. \quad (17)$$

Let  $\mathcal{H}$  be defined by (15).

**LEMMA 2.5.** *If  $\varkappa$  is symplectic (all the points of  $\mathfrak{P}$  are selfconjugate) then  $\mathcal{H}$  determined by a function  $\mathfrak{h}$  defined in (16), (17) coincides with the set  $\bigcup\{x + \mathfrak{h}(x) : x \in S\}$ .*

PROOF. Evident, as  $x \in \varkappa(x)$  yields, in accordance with the definition,  $2x = x + x \in \mathcal{H}$ , for each point  $x$  of  $\mathfrak{P}$ .  $\square$

**LEMMA 2.6.** *Let  $\varkappa$  be symplectic.  $\mathcal{H}$  is a proper subspace of  $\mathfrak{M}$  and therefore it is a hyperplane of  $\mathfrak{M}$ .*

PROOF. Let  $L = a + L_0$  with  $L_0 \in \mathcal{L}$ ; suppose  $|L \cap \mathcal{H}| \geq 2$ . Then there are at least two points  $a + x_1, a + x_2$  in  $L \cap \mathcal{H}$  and either  $a \neq x_1, x_2$  and thus  $x_1, x_2 \in \varkappa(a)$ , which gives  $L_0 \subset \varkappa(a)$ . Consequently,  $L \subset \mathcal{H}$ . Or  $a = x_1 \neq x_2$  and then  $x_2 \in \varkappa(a)$ ; if  $\varkappa$  symplectic then  $x_1 \in \varkappa(a)$  as well and the claim follows.  $\square$

Now, we are in a position to characterize all the hyperplanes in Veronese spaces of level two associated with projective spaces.

**THEOREM 2.7.** *The set  $\mathcal{H}$  is a hyperplane in  $\mathbf{V}_2(\mathfrak{P})$  iff  $\mathcal{H}$  is defined by (15), where  $\mathfrak{h}$  is defined by (16), (17) for some symplectic correlation  $\varkappa$ .*

PROOF. The right-to-left implication follows directly from 2.6. Let  $\mathcal{H}$  be a hyperplane of  $\mathbf{V}_2(\mathfrak{P})$ . Consider the binary relation  $\perp$  on the points of  $\mathfrak{P}$  defined by the condition  $x \perp y \iff x + y \in \mathcal{H}$ . Clearly,  $\perp$  is symmetric. Set  $\mathfrak{h}(x) = \{y : x \perp y\}$  for each point  $x$  of  $\mathfrak{P}$ . By definition,  $x + \mathfrak{h}(x) = (x + S) \cap \mathcal{H}$  is  $\mathfrak{l}$ -transversal in  $x + S$ , so  $\mathfrak{h}(x) = S$  or  $\mathfrak{h}(x)$  is a hyperplane in  $\mathfrak{P}$ . As  $\mathcal{H}$  is a proper subspace, for at least one  $x$  we have  $\mathfrak{h}(x) \neq S$ . From [10] we deduce that  $\perp$  can be characterized by the formula  $\langle u \rangle \perp \langle v \rangle \iff \xi(u, v) = 0$  for a sesquilinear form  $\xi$  defined on  $\mathbb{V}$ . Let  $\varkappa$  be the correlation of  $\mathfrak{P}$  determined by  $\xi$ . Note that  $\mathfrak{h}_2(S) = \bigcup\{x + S : x \in S\}$ , so  $\mathcal{H} = \bigcup\{x + \mathfrak{h}(x) : x \in S\}$  i.e.  $\mathcal{H}$  is defined by (15) with  $\mathfrak{h}$  satisfying (16). Recall that  $\mathfrak{h}(0) = \{x \in S : 2x \in \mathcal{H}\}$  From 2.4 we deduce (formally) that either  $\mathfrak{h}(0) = S$  or  $\mathfrak{h}(0) = h_0$  is a hyperplane and this hyperplane is contained in the set of  $\varkappa$ -selfconjugate points. In both cases we conclude with  $2S \subset \mathcal{H}$ , which gives  $a \perp a$  for each  $a$ . Consequently,  $\varkappa$  is symplectic.  $\square$

So, from now on we assume that  $\varkappa$  is symplectic. Denote

$$\mathfrak{A} := \mathfrak{M} \setminus \mathcal{H}$$

the corresponding affine reduct. From 2.6 it follows that  $\mathfrak{A}$  is a partial affine partial line space.

**LEMMA 2.8.** *The point set of  $\mathfrak{A}$  consists of all the multisets  $x + y$  with  $x, y \in S$  and  $x \notin \kappa(y)$  (so:  $x \neq y$ ); therefore this point set can be identified with a subset of  $\wp_2(S)$ .*

**LEMMA 2.9.** *The hyperplane  $\mathcal{H}$  determined by a symplectic polarity is spiky, but it is not flappy.*

PROOF. Take a point  $e = x + y$  with  $x \perp y$ ,  $x \neq y$ . Then  $y \in x^\perp$ . There is a line  $L_0$  through  $y$  not contained in  $x^\perp$ . Set  $L = x + L_0$ , then  $e \in L$  and  $L$  not contained in  $\mathcal{H}$ . Indeed, suppose  $x + z \in \mathcal{H}$  for  $y \neq z \in L_0$ ; then  $z \in x^\perp$ , so  $L_0 \subset x^\perp$ .

Take any line  $L_0$  of  $\mathfrak{M}$ ; then  $L = 2L_0 \subset \mathcal{H}$ . On the other hand any plane of  $\mathfrak{M}$  which contains  $2L_0$  is contained in  $T(L) = 2S \subset \mathcal{H}$  which yields our second claim.  $\square$

It is rather easy to observe the following

**FACT 2.10.** *The maximal strong subspaces of  $\mathfrak{M}$  are the leaves of  $\mathfrak{M}$ . Consequently, the maximal strong subspaces of  $\mathfrak{A}$  are affine spaces of the form  $(x + S) \setminus \kappa(x)$ ,  $x \in S$  (cf. [4, Fact 2.2], [12, Prop. 2.11]).*

As a direct consequence of 2.10 and 1.6 (cf. [13, Lem. 2.4]) we obtain

**LEMMA 2.11.** (i) *Each Veblen subconfiguration ‘with diagonals’ (i.e. each projective quadrangle) of  $\mathfrak{M}$  is contained in a leaf.*

(ii) *The relation of Veblen parallelism is properly definable in  $\mathfrak{A}$  by the formula (10), i.e. the lines are in the relation  $\parallel^\circ$  when they are on a (affine) plane and are parallel on that plane.*

As an important by-product of 1.6 we get nearly immediately

**PROPOSITION 2.12.** *The structure  $\mathfrak{A}$  satisfies the Tamaschke Bedingung and the Parallelogram Completion Condition.*

The framework proposed admits some degenerations. Namely, we cannot expect that  $\kappa$  is nondegenerate, and thus it may happen that  $\mathcal{H} \supset 2S$  contains some leaves of the form  $x + S$  as well. And then also lines contained in these leaves cannot be extended to “proper” planes of  $\mathfrak{A}$  and  $\mathcal{H}$  is ‘more non-flappy than expected’. In essence, this happens in every odd dimension of  $\mathbb{V}$ .

In what follows we assume that  $\kappa$  is nondegenerate, and then  $\dim(\mathbb{V})$  is even. Let us examine the structure of the parallelism  $\parallel_{\mathcal{H}}$  and of the horizon determined by it. We begin with an evident observation

**LEMMA 2.13.** *Let  $e \in \mathcal{H}$ . Set  $\mathfrak{A} = \mathfrak{M} \setminus \mathcal{H}$ . Then one of the following two possibilities holds*

- For any two lines  $L_1, L_2$  of  $\mathfrak{A}$  which pass through  $e$  there is a plane of  $\mathfrak{A}$  which contains them; in consequence,  $L_1 \parallel^\circ L_2$ : this happens when  $e = 2x$  with  $x \in S$ . In this case the two leaves of  $\mathfrak{M}$  through  $e$  are  $x + S$  and  $2S \subset \mathcal{H}$  and thus  $L_1, L_2$  are determined by two lines, both two in  $x + S$ .

- There are two lines  $L_1, L_2$  through  $e$  such that  $L_1 \parallel^\circ L_2$  and any  $L_3$  through  $e$  (i.e. any  $L_3$  with  $L_3 \parallel L_1$ ) satisfies  $L_3 \parallel^\circ L_1$  or  $L_3 \parallel^\circ L_2$ : *this happens when  $e = x + y$  with  $x \neq y$ , we take  $L_1 \subset x + S$ ,  $L_2 \subset y + S$  (comp. 2.9).*

This allows us to distinguish two types of directions in  $\mathfrak{A}$ ; directions of the first type ( $[L]_\parallel$  when  $L \parallel L'$  iff  $L \parallel^\circ L'$  for each line  $L'$ ) correspond to the elements of one totally deleted leaf  $2S$ . In any case, a direction of  $\mathfrak{A}$  uniquely corresponds to a point in  $\mathcal{H}$ ; in what follows we shall frequently identify corresponding two objects.

For a direction  $\mathfrak{a}$  (an equivalence class under a *parallelism*  $\parallel'$ ) and a set of points  $X$  write

$\mathfrak{a} \mid X$  (in words:  $\mathfrak{a}$  is incident with  $X$ ) when  $\mathfrak{a} = [L]_\parallel'$  for a line  $L \subset X$ .

**Remark 4.** Note that though the two relations  $\parallel$  and  $\parallel^\circ$  *do not coincide*, the relation  $\parallel^\circ$  *is a partial parallelism* so, it also determines its directions (equivalence classes). And, formally speaking, a point  $x + y \in \mathcal{H} \setminus 2S$  determines *two distinct*  $\parallel^\circ$ -directions.

Let us stress on the fact that the distinction formulated in 2.13 refers entirely to  $\parallel_{\mathcal{H}}$ -directions; loosely speaking  $\mathfrak{a} \in 2S$  iff it is incident with exactly one leaf, and other  $\parallel$ -directions are incident with two leaves. But *each  $\parallel^\circ$ -direction is incident with exactly one leaf!*  $\bigcirc$

We pass over the following, interesting on its own right

**PROBLEM 2.14.** Characterize the geometry of the partial affine partial line space  $\langle S', \mathcal{L}', \parallel^\circ \rangle$ , where  $\langle S', \mathcal{L}', \parallel_{\mathcal{H}} \rangle = \mathfrak{A}$  i.e. of the reduct  $\mathbf{V}_2(\mathfrak{P})$  equipped with the parallelism imitating the affine one.  $\bigcirc$

$\mathcal{H}$  is spiky, so its points can be identified with the equivalence classes of  $\parallel_{\mathcal{H}}$ . However,  $\mathcal{H}$  is not flappy, so lines on  $\mathcal{H}$  cannot be identified, in general, with directions of planes of  $\mathfrak{A}$  and the standard way to recover  $\mathfrak{M}$  from  $\mathfrak{A}$  fails. This recovering is still possible though, only the recovering procedure must be complicated a bit.

**LEMMA 2.15.** *The lines of the horizon of  $\mathfrak{A}$  that are contained in a leaf of the form  $x + S$  are definable in  $\mathfrak{A}$ .*

PROOF. Firstly, we note that the class of planes of  $\mathfrak{A}$  can be defined in  $\mathfrak{A}$ . Indeed, let  $\Delta$  be a triangle with the sides  $L_1, L_2, L_3$  and the vertices  $e_1, e_2, e_3$ ,  $e_i \not\parallel L_i$  for  $i = 1, 2, 3$ , such that  $e_1 \sim e_0 \mid L_1$  for some  $e_0 \neq e_2, e_3$ . Then the set

$$\pi(L_1, L_2, L_3) := \bigcup \{L : L \parallel L_1 \wedge L \sim L_2, L_3\} \quad (18)$$

is a plane in  $\mathfrak{A}$  i.e.  $\pi(L_1, L_2, L_3) = a + A$  for a point  $a$  of  $\mathfrak{P}$  and a plane  $A$  of the affine reduct  $\mathfrak{P} \setminus \mathfrak{h}(a)$ , and each plane of  $\mathfrak{A}$  has a form as in (18). So, let  $\mathcal{P}$  be the class of planes. The collinearity of the required form is defined by

$$\mathbf{L}([L_1]_\parallel, [L_2]_\parallel, [L_3]_\parallel) \iff (\exists A \in \mathcal{P})(\exists L'_1, L'_2, L'_3 \subset A) [\wedge_{i=1}^3 L'_i \parallel L_i]. \quad (19)$$

This argument closes the reasoning.  $\square$

**Remark 5.** In the ordinary affine geometry the formula (18) defines a plane for every triangle  $L_1, L_2, L_3$ . In the case of Veronese spaces we must be cautious. Indeed, *if lines  $L_0, L_1, L_2, L_3$  yield in  $\mathfrak{M}$  a Veblen figure of the form 1.5(ii) or 1.5(iii) then  $\pi(L_1, L_2, L_3) = \mathfrak{v}_2(m)$  and, clearly, the latter is not a plane.*  $\bigcirc$

Though the lines on  $2S$  are not “improper lines” of affine planes, we can recover also these lines in terms of  $\mathfrak{A}$ .

**LEMMA 2.16.** *The lines of the horizon of  $\mathfrak{A}$  that are contained in the leaf  $2S$  are definable in terms of  $\mathfrak{A}$ .*

PROOF. Let  $\mathcal{S}$  stand for the class of the maximal strong subspaces of  $\mathfrak{A}$ ; it is definable. From 2.10  $\mathcal{S} = \{a + (S \setminus \kappa(a)) : a \in S\}$ . So, with each line  $L$  of  $\mathfrak{A}$  we have a definable set  $T(L)$  with  $L \subset T(L) \in \mathcal{S}$ . In particular, the notion of a proper quadrangle can be expressed in terms of  $\mathfrak{A}$ . We have for  $\mathfrak{a}_1, \mathfrak{a}_2, \mathfrak{a}_3 \in 2S$

$$\begin{aligned} L(\mathfrak{a}_1, \mathfrak{a}_2, \mathfrak{a}_3) \iff \exists L_1, L_2, L_3 \Big( \exists L', L'', M', M'' : - \text{ a proper quadrangle in } \mathfrak{A} \\ (L_1, L_2, L_3 \sim L', L'' \wedge \bigwedge_{i=1}^3 \mathfrak{a}_i \mid T(L_i)) \Big). \end{aligned} \quad (20)$$

The claim is immediate after 1.8 and 1.9.  $\square$

As an important consequence we obtain now

**THEOREM 2.17.** *The underlying Veronese space  $\mathfrak{M}$  can be recovered from its affine reduct  $\mathfrak{A}$ .*

**PROBLEM 2.18.** Does 2.17 remain true for degenerate  $\kappa$ ?

The claim of 2.17 seems valid also when  $\dim(\text{Rad}(\kappa)) = 1$ . In that case the proof of 2.16 can be modified so as it works in case when  $a + S \subset \mathcal{H}$  for a point  $a \in \text{Rad}(\kappa)$ . However, when  $m$  is a line of  $\mathfrak{P}$  contained in  $\mathcal{H}$  then no line  $a + m$  of  $\mathfrak{A}$  can be completed to a net outside  $\mathcal{H}$  (cf. 1.9) and to prove an analogue of 2.16 one must find some other methods. On the other hand, when  $a \in \text{Rad}(\kappa)$  then each line of  $\mathfrak{M}$  through  $2a$  is contained in  $\mathcal{H}$  and thus  $\mathcal{H}$  is not spiky, so new difficulties arise and no end in sight.

**NOTE 2.** Slightly rephrasing the proof of 2.16 with the help of 1.10 one can prove that *the parallelism  $\parallel_{\mathcal{H}}$  can be defined in terms of the incidence structure of  $\mathfrak{A}$* . Indeed, two lines are parallel either when they are in one leaf: then their parallelism coincides with  $\parallel^\circ$ . Or they are in distinct leaves of  $\mathfrak{M}$ : then they can be completed to a proper net; if they do not intersect, their common point must lie in  $\mathcal{H}$ .

As we already noted, the leaves of  $\mathfrak{A}$  carry the structure of an affine space. Particularly, if  $\mathfrak{P} = PG(n, q)$  ( $2 \nmid n, q$ ) then the leaves of  $\mathfrak{A}$  have the structure of  $AG(n, q)$ . But the geometry of affine reducts of Veronese spaces associated with projective spaces and the geometry of Veronese spaces associated with affine spaces are essentially distinct.

**THEOREM 2.19.** *Let  $\mathcal{H}$  be a hyperplane in  $V_2(\mathfrak{P})$ . Then*

$$V_2(\mathfrak{P}) \setminus \mathcal{H} \not\cong V_k(\mathfrak{A}_0)$$

*for every affine space  $\mathfrak{A}_0$  and every integer  $k$ .*

PROOF. Let  $x + S$  be a leaf of  $\mathfrak{M} = \mathbf{V}_2(\mathfrak{P})$ . Then the corresponding leaf of  $\mathfrak{A} = \mathfrak{M} \setminus \mathcal{H}$  is  $S' := (x + S) \setminus \mathcal{H}$ ; let  $\mathfrak{A}'_0$  be the restriction of  $\mathfrak{A}$  to  $S'$ . Each leaf of  $\mathfrak{A}$  is isomorphic to  $\mathfrak{A}'_0$ . And there pass exactly two leaves through each point of  $\mathfrak{A}$ . Suppose an isomorphism exists. Then  $k = 2$  and  $\mathfrak{A}_0 = \mathfrak{A}'_0$ . To close the proof it suffices to observe that the Veronese Space  $\mathbf{V}_2(\mathfrak{A}_0)$  associated with an affine space satisfies the Net Condition. The reduct  $\mathfrak{A}$  does not satisfy this condition: two lines through  $x + y \in \mathcal{H}$ , one in  $x + S$  and the second in  $y + S$  can be completed to a net on its proper points, and clearly they do not cross each other in  $\mathfrak{A}$ .  $\square$

A PROOF FOR ENTHUSIASTS OF FINITE GEOMETRIES. Let  $\mathfrak{P} = PG(n, q)$ . Comparing the parameters we see that if a suitable affine space exists then  $k = 2$  and  $\mathfrak{A}_0 = AG(n, q)$ . Recall that  $PG(n, q)$  has  $\binom{n+1}{1}_q = \frac{q^{n+1}-1}{q-1}$  points. Since the number of points in  $\mathfrak{A}$  is the number of leaves times number of proper points in a leaf divided on two we obtain

$$\begin{aligned} v_{\mathfrak{A}} &= \frac{1}{2} |PG(n, q)| \cdot |PG(n-1, q)| \\ &= \frac{(q^{n+1}-1)}{2(q-1)} q^n. \end{aligned}$$

Note that since symplectic polarities exist in even dimensions only, we have  $n = 2m - 1$  for some integer  $m \geq 2$ . Then  $q^{n+1} - 1 = q^{2m} - 1 = (q^m - 1)(q^m + 1)$ . Then  $2|(q^m + 1)$  and  $(q - 1)|(q^m - 1)$ . The same formula for  $v_{\mathfrak{A}}$  applied to the case  $n = 2$ ,  $q = 3$  gives a proper fraction!

The number of points in  $\mathfrak{A}' = \mathbf{V}_2(AG(n, q))$  is

$$\begin{aligned} v_{\mathfrak{A}'} &= \binom{|AG(n, q)| + 1}{2} \\ &= \frac{1}{2} (q^n + 1)(q^n). \end{aligned}$$

With the help of MapleV we compute that

$$\varepsilon(n, q) = v_{\mathfrak{A}} - v_{\mathfrak{A}'} = \frac{q^{2n} - q^{n+1}}{2(q-1)} = \frac{q^n(q^n - q)}{2(q-1)} > 0.$$

Substituting we get, e.g. that  $\varepsilon(3, 3) = 162 > 0$ .  $\square$

### 2.2.2 Generalization: hyperplanes in Veronesians with level $k > 2$ associated with projective spaces

The construction of a hyperplane determined by a symplectic form can be applied to Veronesians of the level greater than 2 as well. Let  $\eta: V^k \rightarrow F$  be a  $k$ -linear nondegenerate symplectic form, defined on a vector space  $\mathbb{V}$  with  $V$  being its set of vectors, let  $F$  be the set of scalars. Recall two basic properties of  $\eta$ :

- a)  $\eta(v_1, \dots, v_k) = 0$  yields  $\eta(v_{\sigma(1)}, \dots, v_{\sigma(k)}) = 0$  for every permutation  $\sigma$  of  $\{1, \dots, k\}$ .
- b) if  $v_i = v_j$  for  $1 \leq i < j \leq k$  then  $\eta(v_1, \dots, v_k) = 0$ .

For a family  $q_1 = \langle v_1 \rangle, \dots, q_k = \langle v_k \rangle$  of points of  $\mathfrak{P} = \mathbf{P}_1(\mathbb{V}) = \langle S, \mathcal{L} \rangle$  we write  $\perp_\eta(q_1, \dots, q_k)$  when  $\eta(v_1, \dots, v_k) = 0$ . From the property a) of  $\eta$  we get that the relation  $\perp = \perp_\eta$  is fully symmetric.

Define

$$\mathcal{H} := \{q_1 + q_2 + \dots + q_k : q_1, \dots, q_k \text{ - points of } \mathfrak{P}, \perp(q_1, \dots, q_k)\} \quad (21)$$

**FACT 2.20.** *The set  $\mathcal{H}$  defined by (21) is a (nondegenerate) hyperplane in  $\mathbf{V}_k(\mathfrak{P})$ .*

From the property b) of  $\eta$ , the point universe of  $\mathbf{V}_k(\mathfrak{P}) \setminus \mathcal{H}$  i.e. the set  $\mathfrak{p}_k(S) \setminus \mathcal{H}$  is a subset of  $\mathcal{P}_k(S)$  and therefore the affine reduct  $\mathbf{V}_k(\mathfrak{P}) \setminus \mathcal{H}$  is a substructure of  $\mathbf{G}_k(\mathfrak{P})$ . Happily, this substructure does not admit deviations presented in 1.11: if  $\kappa$  is the size of lines of  $\mathfrak{P}$  then the size of lines of  $\mathbf{V}_k(\mathfrak{P}) \setminus \mathcal{H}$  is  $\kappa - 1$ . Passing over the general theory of affine reducts we can establish this equation directly from 1.11 and (7). Indeed, if  $a = \{q_1, \dots, q_{k-1}\}$  is a  $(k-1)$ -element subset of  $S$  and  $L$  is a line of  $\mathfrak{P}$  then either  $|a^\perp \cap L| \geq 2$  and then  $a + L \subset \mathcal{H}$  or  $|a^\perp \cap L| = 1$  and then at most one of the elements of  $a$  is in  $L$ .

So, one can characterize directly the structure  $\mathfrak{A} = \mathbf{V}_k(\mathfrak{P}) \setminus \mathcal{H}$  as a substructure of  $\mathbf{G}_k(\mathfrak{P})$ :

the points of  $\mathfrak{A}$ :  $\{\{q_1, \dots, q_k\} \in \mathcal{P}_k(S) : \not\perp(q_1, \dots, q_k)\}$ ,

the lines of  $\mathfrak{A}$ :  $\{\{\{q_1, \dots, q_{k-1}, x\} : x \in L, \not\perp(q_1, \dots, q_{k-1}, x)\} : L \in \mathcal{L}, L \not\subset (q_1, \dots, q_{k-1})^\perp\}$ .

The structures obtained are more ‘Grassmannians’ than ‘Veronesians’: they are defined on sets *without repetitions*! The problem to enter deeper into geometry of such reducts of Grassmannians is addressed in another papers. Let us indicate only one connection with some other celebrated geometry: Let  $u_1, \dots, u_k$  be vectors of  $\mathbb{V}$ . The set  $\{\langle u_1 \rangle, \dots, \langle u_k \rangle\} \in \mathcal{P}_k(S)$  is a point of the affine reduct  $\mathfrak{A}$  iff vectors  $u_1, \dots, u_k$  are linearly independent i.e. when they form a basis of a  $k$ -subspace of  $\mathbb{V}$ . So,  $\mathfrak{A}$  is the structure of (unordered) bases of  $\mathfrak{P}$ . Moreover, the map

$$\text{a point } \{q_1, \dots, q_k\} \text{ of } \mathfrak{A} \longmapsto \overline{\{q_1, \dots, q_k\}} : \text{a } k\text{-subspace of } \mathbb{V}$$

is a monomorphism of  $\mathfrak{A}$  onto the Grassmann space (space of pencils)  $\mathbf{P}_k(\mathbb{V}) \cong \mathbf{P}_{k-1}(\mathfrak{P})$ .

### 2.2.3 More sophisticated example: Veronese spaces associated with polar spaces

Let us begin with three rather evident observations

**FACT 2.21.** *Let  $\mathcal{H}$  be a hyperplane in a partial linear space  $\mathfrak{N} = \langle S, \mathcal{L} \rangle$ , let  $S_0 \subset S$ . Set  $\mathcal{L}[S_0] = \{L \in \mathcal{L} : L \subset S_0\}$ . and let  $\mathcal{L}_0 \subset \mathcal{L}[S_0]$ . Then  $\mathcal{H} \cap S_0$  is an  $l$ -transversal subset in  $\langle S_0, \mathcal{L}_0 \rangle$ .*

In the notation of the fact 2.21 we write  $\mathfrak{N}[S_0] = \langle S_0, \mathcal{L}[S_0] \rangle$ . Generally,  $\mathfrak{N}[S_0]$  needs not be a partial linear space, but for some ‘nonsense-reasons’ only: its line set may be empty, it may have isolated points etc.

**FACT 2.22.** *Let  $\mathfrak{M}_0 = \langle S_0, \mathcal{L}_0 \rangle$  be a partial linear space and let  $S'_0 \subset S_0$ . Then  $\mathbf{V}_k(\langle S'_0, \mathcal{L}[S'_0] \rangle)$  and  $\mathbf{V}_k(\langle S_0, \mathcal{L}_0 \rangle)[\mathfrak{p}_k(S'_0)]$  coincide.*



PROOF. The point sets of both structures are equal: just from definitions.

Let  $L$  be a line of  $\mathbf{V}_k(\langle S'_0, \mathcal{L}[S'_0] \rangle)$ . So,  $L = e + (k - |e|)L_0$ , where:  $L_0 \in \mathcal{L}[S'_0] \subset \mathcal{L}_0$ , so  $L_0 \subset S'_0$ , and  $e \in \mathfrak{h}_k(S'_0) \subset \mathfrak{h}_k(S_0)$ . Finally,  $L$  is a line of  $\mathbf{V}_k(\langle S_0, \mathcal{L}_0 \rangle)$  and  $L \subset \mathfrak{h}_k(S'_0)$ . Conversely, let  $L = e + (k - |e|)L_0$  be a line of  $\mathbf{V}_k(\langle S_0, \mathcal{L}_0 \rangle)$ . Assume that  $L \subset \mathfrak{h}_k(S'_0)$ . Then  $e \in \mathfrak{h}_k(S'_0)$  and  $L_0 \subset S'_0$ ,  $L_0 \in \mathcal{L}_0$  and thus  $L$  is a line of  $\mathbf{V}_k(\langle S'_0, \mathcal{L}[S'_0] \rangle)$ .  $\square$

**FACT 2.23.** *If  $\mathfrak{M}' = \langle S, \mathcal{L}' \rangle$  and  $\mathfrak{M}'' = \langle S, \mathcal{L}'' \rangle$  are partial linear spaces such that  $\mathcal{L}'' \subset \mathcal{L}'$  then the line set of  $\mathbf{V}_k(\mathfrak{M}'')$  is a subset of the line set of  $\mathbf{V}_k(\mathfrak{M}')$ .*

Let us quote the standard models of polar spaces (in what follows a *polar space* will always mean one of the below). Let  $\varpi$  be a polarity in  $\mathfrak{P} = \langle S, \mathcal{L} \rangle$ ; let  $Q_0(\varpi) = \{p: p \in \varpi(p)\}$  be the set of points of  $\mathfrak{P}$  that are self-conjugate under  $\varpi$  and  $Q_1(\varpi) = \{L: L \subset \varpi(L)\}$  be the set of selfconjugate lines. Then the polar space determined by  $\varpi$  is the structure  $\mathbf{Q}(\varpi) := \langle Q_0(\varpi), Q_1(\varpi) \rangle$  provided  $\varpi$  is symplectic or  $\varpi$  is quadratic with index at least 2 i.e.  $Q_2(\varpi) \neq \emptyset$ . In corresponding cases we have  $\mathbf{Q}(\varpi) = \langle S, Q_1(\varpi) \rangle$  and  $\mathbf{Q}(\varpi) = \mathfrak{P}[Q_0(\varpi)]$ . Let us recall also that an *affine polar space* is an affine reduct of a polar space obtained by deleting a hyperplane of it. In our approach this hyperplane can be always considered as the restriction to  $Q_0$  of a hyperplane of  $\mathfrak{P}$ .

The following will be needed, that is known in the literature.

**FACT 2.24.** *A maximal strong subspace of a polar space (of an affine polar space) is a projective (affine, resp.) subspace of  $\mathfrak{P}$  (of the affine reduct of  $\mathfrak{P}$ ).*

*Any two planes of a polar space and of an affine polar space can be joined by a sequence of planes each two consecutive planes in the sequence sharing a line (a polar space and an affine polar space is strongly connected, cf. [11]).*

Let us consider the structure  $\mathfrak{M} := \mathbf{V}_k(\mathbf{Q}(\varpi))$ . Then as an immediate consequence of 2.23 and 2.22 resp. we have

**LEMMA 2.25.**

- (i) *When  $\varpi$  is symplectic, the line set of  $\mathfrak{M}$  is a subset of the line set of  $\mathbf{V}_k(\mathfrak{P})$ ,*
- (ii)  *$\mathfrak{M} = \mathbf{V}_k(\mathfrak{P})[\mathfrak{h}_k(Q_0(\varpi))]$  when  $\varpi$  is quadratic.*

Let  $\mathcal{H}$  be a hyperplane of  $\mathbf{V}_k(\mathfrak{P})$  determined by a polarity  $\varkappa$ . From 2.21 and 2.25 we obtain

**FACT 2.26.** *The set  $\mathcal{H} \cap \mathfrak{h}_k(Q_0(\varpi))$  is a hyperplane in  $\mathfrak{M}$ .*

In the sequel we write simply  $\mathfrak{M} \setminus \mathcal{H}$  instead of  $\mathfrak{M} \setminus (\mathcal{H} \cap \mathfrak{h}_k(Q_0(\varpi)))$ .

**LEMMA 2.27.** *The leaves of  $\mathfrak{M}$  and their restrictions in  $\mathfrak{M} \setminus \mathcal{H}$  are definable in the internal geometry of  $\mathfrak{M}$  (of  $\mathfrak{M} \setminus \mathcal{H}$ , resp.).*

PROOF. In the first step we consider strong subspaces of  $\mathfrak{M}$  and  $\mathfrak{M} \setminus \mathcal{H}$ . Clearly, they have form  $e + (k - |e|)X$  ( $(e + (k - |e|)X) \setminus \mathcal{H} = e + (k - |e|)A$ , resp.), where  $X$  is a strong subspace of  $\mathbf{Q}(\varpi)$  ( $A$  is a strong subspace of the affine polar space  $\mathbf{Q}(\varpi) \setminus \varkappa(e)$  resp.).

Therefore, in the second step we can define the class  $\mathcal{P}$  of planes of  $\mathfrak{M}$  (of  $\mathfrak{M} \setminus \mathcal{H}$ ).

In the third step we define a point-to-point relation  $\gamma$ :  $a \gamma b$  when  $a$  and  $b$  can be joined by a sequence of elements of  $\mathcal{P}$  each two consecutive planes sharing a line.

In the last step we note that the leaves in question are the equivalence classes under the relation  $\gamma$ .  $\square$

Let  $\varkappa$  be symplectic and  $k = 2$ . Applying the techniques of the previous subsection we can prove now

**THEOREM 2.28.** *The underlying Veronese space  $\mathfrak{M}$  can be defined in terms of its affine reduct  $\mathfrak{M} \setminus \mathcal{H}$ .*

**PROBLEM 2.29.** The case  $\varkappa = \varpi$  seems the most interesting, regular, and intriguing. How to characterize it?  $\circ$

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